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Local functional equations associated with decomposable graphs

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Abstract

Let G be a graph with n vertexes $\{v_1, \ldots, v_n\}$ (without multiple edges) and consider the vector space

$$\mathrm{Sym}_G(\mathbb{R}) = \{X \in \mathrm{Sym}_n(\mathbb{R}) | X_{ij} = 0 \ (v_i \not\sim v_j)\}.$$

Denote by $\mathrm{Sym}_G^*(\mathbb{R})$ its dual vector space. With a statistical motivation, Letac and Massam (Ann. of Statistics, 2007) calculated explicitly the Gamma integral attached to the cones of positive definite "matrices" in $\mathrm{Sym}_G^*(\mathbb{R})$ and the dual cone in $\mathrm{Sym}_G(\mathbb{R})$ under the condition that G is decomposable. From their result we can derive rather easily the functional equation for the local zeta functions attached to the cones. In this note, we report that the local zeta functions attached to not necessarily definite connected components also satisfy functional equations. The cones for decomposable G are in general not homogeneous and our functional equations can not be obtained from the theory of prehomogeneous vector spaces. Proofs will appear elsewhere.

1 Gamma integrals and local functional equations

1.1 We begin by a motivating example. Let $\mathcal{P}_n$ be the cone of positive definite symmetric matrices of size n. Then it is well-known that the following formula holds for $T \in \mathcal{P}_n$:

$$\int_{\mathcal{P}_n} (\det X)^s e^{-\mathrm{tr}(XT)} \frac{dX}{(\det X)^{(n+1)/2}} = \Gamma_n(s)(\det T)^{-s}, \tag{1}$$

$$\Gamma_n(s) = \pi^{n(n-1)/4} \prod_{i=0}^{n-1} \Gamma\left(s - \frac{i}{2}\right).$$

The function $\Gamma_n(s)$ is known as the gamma function of the cone $\mathcal{P}_n$ and we call the integral on the left-hand side of (1) the *Gamma integral* (of $\mathcal{P}_n$). The gamma integral was first considered by Wishart ([Wi]) in a context of statistics and later by C. L. Siegel ([Si]) in the arithmetic of quadratic forms.

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The identity (1) can be continued to $\mathcal{P}_n + \sqrt{-1}\mathrm{Sym}_n(\mathbb{R})$ and we have

$$\int_{\mathcal{P}_n} (\det X)^s e^{-\mathrm{tr}(XZ)} \frac{dX}{(\det X)^{(n+1)/2}} = \Gamma_n(s)(\det Z)^{-s} \tag{2}$$
$$(Z = T + \sqrt{-1}Y \in \mathcal{P}_n + \sqrt{-1}\mathrm{Sym}_n(\mathbb{R})).$$

For $s \in \mathbb{C}$ with $\Re(s) > 0$, denote by $|\det Y|_j^s$ $(j = 0, 1, \ldots, n)$ the continuous function defined by

$$|\det Y|_j^s = \begin{cases} |\det Y|^s, & \text{if } Y \text{ has } j \text{ positive and } n-j \text{ negative eigenvalues,} \\ 0, & \text{otherwise.} \end{cases}$$

The functions $|\det Y|_j^s$ are continued to tempered distributions with meromorphic parameter s to the whole complex plane. We denote the meromorphic continuations by the same symbol $|\det Y|_j^s$. Then by taking the limit of (2) as $Z = T + \sqrt{-1}Y \to \sqrt{-1}Y$, we can obtain the following formula for the Fourier transform of $|\det X|_n^s$:

$$\int_{\mathrm{Sym}_n(\mathbb{R})} |\det X|_n^{s-(n+1)/2} \, e^{2\pi\sqrt{-1}\mathrm{tr}(XY)} dX$$
$$= (2\pi)^{-ns}\Gamma_n(s) \sum_{j=0}^{n} e^{s(2j-n)\pi\sqrt{-1}/2} \, |\det Y|_j^{-s}. \tag{3}$$

1.2 It is natural to expect similar formulas hold for the Fourier transforms of the distributions $|\det X|_i^s$ whose supports are contained in the set of indefinite symmetric matrices. The existence of such formulas is assured by the theory of prehomogeneous vector spaces ([SS]) and the explicit formula was given by Shintani ([Sh]). We put

$$(\sqrt{-1})^{-l(l+1)/2}\xi_{l,m} = \begin{cases} (-1)^{(l-m)/2}\binom{l/2}{m/2} & (l \equiv 0,\ m \equiv 0 \pmod 2), \\ 0 & (l \equiv 0,\ m \equiv 1 \pmod 2), \\ (-1)^{(l+1-m)/2}\binom{(l-1)/2}{m/2} & (l \equiv 1,\ m \equiv 0 \pmod 2), \\ (-1)^{(l-m)/2}\binom{(l-1)/2}{(m-1)/2} & (l \equiv m \equiv 1 \pmod 2) \end{cases}$$

and

$$\begin{aligned} u_{ij}^{(n)}(s) &= \sqrt{-1}^{(n+1)(i+j-n/2)}(-1)^{(n-j)(n-j+1)/2} \\ &\quad \times \sum_{r=\max(0,i+j-n)}^{\max(i,j)} (-1)^{r(n+1)}\xi_{j,r}\xi_{n-j,i-r}e^{(2r-i-j)\pi s\sqrt{-1}}. \end{aligned} \tag{4}$$

Then Shintani's formula reads

$$\int_{\mathrm{Sym}_n(\mathbb{R})} |\det X|_i^{s-(n+1)/2} \, e^{2\pi\sqrt{-1}\mathrm{tr}(XY)} dX$$
$$= (2\pi)^{-sn}\Gamma_n(s)e^{\pi s\sqrt{-1}/2} \sum_{j=0}^{n} u_{ij}^{(n)}(s) \, |\det Y|_j^{-s}. \tag{5}$$

1.3 We call a formula similar to (5) a *local functional equation*. More precisely, consider two homogeneous polynomials $P(x) = P(x_1, \ldots, x_n)$ and $Q(y) =$

$Q(y_1, \ldots, y_n)$ with real coefficients and of degree d. Let $\Omega_1, \ldots, \Omega_\nu$ (resp. $\Omega_1^*, \ldots, \Omega_{\nu^*}^*$) be the connected components of $\mathbb{R}^n \setminus \{P(x) = 0\}$ (resp. $\mathbb{R}^n \setminus \{Q(y) = 0\}$). We assume that $\nu = \nu^*$. Denote by $|P(x)|_i^s$, $|Q(y)|_i^s$ be the tempered distribution obtained from

$$|P(x)|_i^s = \begin{cases} |P(x)|^s & (x \in \Omega_i), \\ 0 & (x \notin \Omega_i), \end{cases} \qquad |Q(y)|_i^s = \begin{cases} |Q(y)|^s & (y \in \Omega_i^*), \\ 0 & (y \notin \Omega_i^*), \end{cases}$$

by analytic continuation. If these distributions satisfy identities of the form

$$\int_{\mathbb{R}^n} |P(x)|^s \exp(2\pi\sqrt{-1}(x,y))\, dy = \sum_{i=1}^{\nu} \gamma_{ij}(s)\, |Q(y)|^{-n/d-s}$$

for some meromorphic functions $\gamma_{ij}(s)$, then we call the identities *local functional equations*. Since polynomials P, Q having local functional equations are expected to have rich arithmetic and analytic properties, it is interesting to find abundant pairs of such (P, Q). The theory of prehomogeneous vector spaces ([SS], [S1], [K]) gives a method of constructing pairs (P, Q) satisfying local functional equations, which is the only systematic construction known so far. However, it is also known that there exist pairs of polynomials (P, Q) satisfying local functional equations and not obtained from prehomogeneous vector spaces (see [S2], [KS]).

1.4 The argument in §1.2 and §1.3 suggests that, if we could obtain a formula for a multivariate Laplace transform analogous to (1), then we might expect the existence of local functional equations. Guided by a motivation in multivariate statistics (generalization of the Wishart distribution), Letac and Massam [LM] found a multivariable generalization of the identity (1). In the subsequent sections, we explain the multivariable local functional equations which are expected from the result of Letac and Massam. What is interesting is that we can obtain many examples of local functional equations which do not come from prehomogeneous vector spaces. At present we do not have any theoretical explanations which unify the functional equations for prehomogeneous vector spaces, the results of [S2] and [KS], and the result (Theorem 3.2) in the present paper. Characterization of polynomials having local functional equations is an interesting open problem.

2 Preliminaries on decomposable graphs

To describe the result of Letac and Massam, we need some preparation from the graph theory.

2.1 Let $G = (V, E)$ be an unoriented graph. Namely, V is a set of vertices and $E \subset V \times V$ is a set of edges and we assume that

- $(i, i) \in E$ for all $i \in V$,
- $(i, j) \in E \Longrightarrow (j, i) \in E$.

From now on we call an unoriented graph satisfying these two conditions simply a graph.

A graph $G_1 = (V_1, E_1)$ is called a *subgraph* of a graph $G_2 = (V_2, E_2)$, if $V_1 \subset V_2$ and $E_1 \subset E_2$. For a subset W of V, we can define the induced subgraph $G_W = (W, E_W)$ $(E_W = E \cap (W \times W))$. We often identify the subgraph G_W with W when there is no fear of confusion. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be subgraphs of a graph $G = (V, E)$. Then the sum $G_1 \oplus G_2$ and the intersection $G_1 \cap G_2$ of G_1 and G_2 are defined to be the graphs $(V_1 \cup V_2, E_1 \cup E_2)$ and $(V_1 \cap V_2, E_1 \cap E_2)$, respectively. Note that $G_1 \oplus G_2$ does not necessarily coincides with the induced subgraph $G_{V_1 \cup V_2}$ even if $G_1 = G_{V_1}$ and $G_2 = G_{V_2}$. A graph $G = (V, E)$ is called *complete*, if $E = V \times V$. A maximal complete subgraph of a graph $G = (V, E)$ is called a *clique* of G.

Let $C_1, \dots, C_k$ be the cliques of a graph G and define H_i $(i = 1, \dots, k)$ by

$$H_1 = C_1, \ H_i = H_{i-1} \oplus C_i \ (i = 2, \dots, k).$$

We call the graph G *decomposable*, if the following two conditions hold for a suitable order of the cliques:

- $G = H_k$,
- for any $i \geq 2$, there exists a j with $1 \leq j \leq i-1$ such that $S_i := H_{i-1} \cap C_i \subset C_j$.

The subgraphs $S_2, \dots, S_k$ defined in the latter condition are called *separators*, which are complete subgraphs. Among $S_2, \dots, S_k$ a same separator S can appear several times. We put $\nu(S) = \sharp \{ i \mid S_i = S \}$ and call it the *multiplicity* of S. An order of the cliques satisfying the above two conditions is called a *perfect* order. A decomposable graph may admit several perfect orders. The separators and their multiplicities do not depend on the choice of perfect order. A practical criterion of decomposability is the following:

> G is decomposable if and only if G contains no cycles of length ≥ 4 as induced subgraphs.

Here we mean by a cycle of length r a graph with r vertices $v_1, \dots, v_r$ and the set of edges

$$\{ (v_i, v_j) \mid i - j \equiv 0, \pm 1 \pmod{r} \}.$$

If a decomposable graph G contains no graphs of type A_4 as induced subgraphs, then, G is called *homogeneous*.

2.2 Let $G = (V, E)$ be a graph with n vertices. We identify the vertices with $\{1, 2, \dots, n\}$. We introduce a vector subspace Sym_G and a quotient space Sym_G^* of the space Sym_n of n by n symmetric matrices.

For $K = \mathbb{R}$ or $\mathbb{C}$, we put

$$\mathrm{Sym}_G(K) = \{ X \in \mathrm{Sym}_n(K) \mid X_{ij} = 0 \text{ if } (i,j) \notin E \}, \tag{6}$$

$$\mathrm{Sym}_G^{\perp}(K) = \{ X \in \mathrm{Sym}_n(K) \mid X_{ij} = 0 \text{ if } (i,j) \in E \}. \tag{7}$$

We also put

$$\mathrm{Sym}_G^*(K) = \mathrm{Hom}_K(\mathrm{Sym}_G(K), K). \tag{8}$$

We identify the dual vector space of $\mathrm{Sym}_n(K)$ with $\mathrm{Sym}_n(K)$ via the symmetric bilinear form $(X, Y) \mapsto \mathrm{tr}(XY)$. Then, the space $\mathrm{Sym}_G^*(K)$ is identified with

$\mathrm{Sym}_n(K)/\mathrm{Sym}_G^{\perp}(K)$ and the (i,j)-entry of $Y \in \mathrm{Sym}_G^*(K)$ is well-defined if $(i,j) \in E$.

Let W be a subset of V (equivalently an induced subgraph of G) and put $X_W = (X_{ij})_{i,j\in W}$ for $X \in \mathrm{Sym}_G(K)$. If the induced subgraph G_W is complete, then $Y_W = (Y_{ij})_{i,j\in W}$ is well-defined for $Y \in \mathrm{Sym}_G^*(K)$.

Lemma 2.1 *Let W be a subset of V such that the induced subgraph G_W is connected. Then, the polynomial* $\det X_W$ $(X \in \mathrm{Sym}_G(K))$ *is irreducible.*

2.3 From now on we assume that G is decomposable and let $C_1, \dots, C_k$ be the cliques of G in perfect order and $S_2, \dots, S_k$ the separators. We consider the polynomial functions $Q_{C_i}(Y) = \det Y_{C_i}$ $(i = 1, \dots, k)$ and $Q_{S_i}(Y) = \det Y_{S_i}$ $(i = 2, \dots, k)$ on $\mathrm{Sym}_G^*(K)$. Since C_i and S_i are complete subgraphs of G, these polynomials are well-defined.

Let $\phi : \mathrm{Sym}_G(\mathbb{C}) \to \mathrm{Sym}_G^*(\mathbb{C})$ be the rational map defined by $\phi(X) = X^{-1} \bmod \mathrm{Sym}_G^{\perp}(\mathbb{C})$. We consider the rational functions $P_{C_i}(X) = Q_{C_i}(\phi(X))$ and $P_{S_i}(X) = Q_{S_i}(\phi(X))$. If we denote $\overline{C}_i$ and $\overline{S}_i$ the complements of C_i and S_i, respectively, in V, then we have

$$P_{C_i}(X) = \frac{\det X_{\overline{C}_i}}{\det X}, \quad P_{S_i}(X) = \frac{\det X_{\overline{S}_i}}{\det X}. \tag{9}$$

Since $\overline{C}_i$ and $\overline{S}_i$ are not necessarily connected, the denominators of $P_{C_i}(X)$ and $P_{S_i}(X)$ may be reducible. It is also known that

$$\det X = \frac{\prod_{i=2}^k P_{S_i}(X)}{\prod_{i=1}^k P_{C_i}(X)}. \tag{10}$$

Put

$$\Omega_G(K) = \left\{ X \in \mathrm{Sym}_G(K) \;\middle|\; \det X \cdot \prod_{i=1}^k \det X_{\overline{C}_i} \cdot \prod_{i=2}^k \det X_{\overline{S}_i} \neq 0 \right\}, \tag{11}$$

$$\Omega_G^*(K) = \left\{ Y \in \mathrm{Sym}_G^*(K) \;\middle|\; \prod_{i=1}^k \det Y_{C_i} \cdot \prod_{i=2}^k \det Y_{S_i} \neq 0 \right\}. \tag{12}$$

The following lemma as well as (10) is essentially proved in [L, Chapter 5].

Lemma 2.2 *The map* $\phi : \mathrm{Sym}_G(\mathbb{C}) \to \mathrm{Sym}_G^*(\mathbb{C})$, $\phi(X) = X^{-1} \bmod \mathrm{Sym}_G^{\perp}(\mathbb{C})$ *is birational and the inverse rational map is given by*

$$\phi^* : \mathrm{Sym}_G^*(\mathbb{C}) \to \mathrm{Sym}_G(\mathbb{C}), \quad \phi^*(Y) = \sum_{i=1}^k \left[(Y_{C_i})^{-1}\right]^0 - \sum_{i=2}^k \left[(Y_{S_i})^{-1}\right]^0,$$

where $\left[(Y_H)^{-1}\right]^0$ $(H = C_i, S_i)$ *denotes the matrix in* $\mathrm{Sym}_G(\mathbb{C})$ *whose (i,j)-entry is equal to that of $(Y_H)^{-1}$ or 0 according as $i, j \in H$ or not. Moreover ϕ and ϕ^* induce isomorphisms of Ω_G and Ω_G^*.*

Note that ϕ and ϕ^* have more suggestive expressions

$$\phi(X) = \operatorname{grad}\log(\det X), \quad \phi^*(Y) = \operatorname{grad}\log\left(\frac{\prod_{i=1}^k \det Y_{C_i}}{\prod_{i=2}^k \det Y_{S_i}}\right).$$

2.4 Denote by $\mathcal{C}$ (resp. $\mathcal{S}$) the set of cliques (resp. separators) of decomposable G. We call a mapping $\sigma : \mathcal{C} \cup \mathcal{S} \to \mathbb{N}_{\geq 0}$ a *signature distribution* for G, if it satisfies that

$$0 \leq \sigma(C) \leq \sharp(C)\ (C \in \mathcal{C}), \quad 0 \leq \sigma(S) \leq \sharp(S)\ (S \in \mathcal{S}), \quad \sigma(S) \leq \sigma(C) \text{ if } S \subset C.$$

Denote by $\mathrm{sgn}(G)$ the set of all signature distributions for G. For a $\sigma \in \mathrm{sgn}(G)$, we put

$$\begin{aligned} \Omega^*_{G,\sigma} &= \{Y \in \Omega^*_G(\mathbb{R}) \mid \mathrm{sgn} Y_H = (\sigma(H), \sharp(H) - \sigma(H))\ (H \in \mathcal{C} \cup \mathcal{S})\}, \\ \Omega_{G,\sigma} &= \phi^{-1}(\Omega^*_{G,\sigma}). \end{aligned}$$

Here we denote by $\sharp(H)$ the number of vertices of H. (Recall the convention that we identify an induced subgraph with the set of its vertices.)

Consider the special case where $\sigma(H) = \sharp(H)$ for any $H \in \mathcal{C} \cup \mathcal{S}$. Then $\Omega_{G,\sigma}$ is the set $\mathrm{Sym}^+_n(\mathbb{R})$ of positive definite symmetric matrices in $\mathrm{Sym}_n(\mathbb{R})$ and $\Omega^*_{G,\sigma}$ is the set $\mathrm{Sym}^{*+}_n(\mathbb{R})$ of elements in $\mathrm{Sym}^*_G(\mathbb{R})$ such that Y_H is positive definite for all $H \in \mathcal{C} \cup \mathcal{S}$.

3 Gamma integrals and local zeta functions for decomposable graphs

3.1 As in the previous section, let G be a decomposable graph and denote by $\mathcal{C}$ and $\mathcal{S}$ the sets of cliques and separators of G, respectively. Fix a prefect order of the cliques $C_1, \ldots, C_k$ and let $S_2, \ldots, S_k$ be the corresponding order of the separators. Put

$$H_G(\alpha, \beta; Y) = \left(\prod_{i=1}^{k} |Q_{C_i}(Y)|^{\alpha_i}\right) \Big/ \left(\prod_{i=2}^{k} |Q_{S_i}(Y)|^{\beta_i}\right).$$

Here, S_i and S_j coincide for different i, j and then we consider $\beta_i = \beta_j$. We define a measure $\mu_G(dY)$ on $\{Y \in \mathrm{Sym}^*_G \mid |Q_H(Y)| \neq 0\ (H \in \mathcal{C} \cup \mathcal{S})\}$ ($\supset \mathrm{Sym}^{*+}_G$) by

$$\mu_G(dY) = H_G\left(-\frac{c_1+1}{2}, \ldots, -\frac{c_k+1}{2}; -\frac{s_2+1}{2}, \ldots, -\frac{s_k+1}{2}; Y\right) dY,$$

where we put $c_i = \sharp(V_{C_i})$, $s_i = \sharp(V_{S_i})$. Lettac and Massam introduced the following generalization of the gamma integral (1):

$$\mathcal{L}_G(\alpha, \beta; X) = \int_{\mathrm{Sym}^{*+}_G} e^{-\langle X, Y\rangle} H_G(\alpha, \beta; Y) \mu_G(dY) \quad (X \in \mathrm{Sym}^+_G).$$

When G is a complete graph, then $\mathcal{L}_G(\alpha, \beta; x)$ coincides with the gamma integral (1).

If the graph G is homogeneous in the sense at the end of §2.1, Sym^{*+}_G is a homogeneous convex cone and the integral $\mathcal{L}_G(\alpha, \beta; x)$ can be calculated by the theory of Vinberg and Gindikin. Moreover $Q_H(Y)$ $(H \in \mathcal{S} \cup \mathcal{C})$ are the fundamental relative invariants of certain prehomogeneous vector space and hence satisfy local functional equations. The point of the paper [LM] is that, even if the graph G is not homogeneous, but decomposable, the integral above can be calculated explicitly under a suitable specialization of parameters $\beta_3, \dots, \beta_k$ (see the condition (1) in Theorem 3.1 below).

For a separator S we put

$$J(S) = \{ j \mid 2 \le j \le k,\ S_j = S \}, \quad \nu(S) = \sharp(J(S)).$$

Thoerem 3.1 (Letac-Massam) *For $\alpha_1, \dots, \alpha_k, \beta_2, \dots, \beta_k \in \mathbb{C}^{2k-2}$ satisfying*

(1) $\beta_i = \frac{1}{\nu(S_i)} \sum_{j \in J(S_i)} \alpha_j \quad (S_i \ne S_2)$,

(2) $\Re(\alpha_i) > \frac{c_i - 1}{2} \quad (1 \le i \le k)$,

(3) $\Re(\alpha_1) + \Re(\delta_2) > \frac{s_2 - 1}{2}, \quad \delta_2 = \sum_{j \in J(P, S_2)} \alpha_j - \nu(S_2)\beta_2$,

the integral $\mathcal{L}_G(\alpha, \beta; x)$ is absolutely convergent and we have

$$\mathcal{L}_G(\alpha, \beta; x) = \Gamma_G(\alpha, \beta) H_G(\alpha, \beta; \phi(x)),$$

$$\Gamma_G(\alpha, \beta) = \Gamma_{s_2}(\alpha_1 + \delta_2) \frac{\Gamma_{c_1}(\alpha_1)}{\Gamma_{s_2}(\alpha_1)} \prod_{q=2}^{k} \frac{\Gamma_{c_q}(\alpha_q)}{\Gamma_{s_q}(\alpha_q)}.$$

Note that, by the assumption (1), only $\alpha_1, \dots, \alpha_k, \beta_2$ are independent complex parameters. We can prove further that there exist $k+1$ irreducible polynomials $P_{G,0}(X) = \det X$, $P_{G,1}(X), \dots, P_{G,k}(X)$ such that $H_G(\alpha, \beta; \phi(x))$ is a complex power of these polynomials (see (14)). The proof of Theorem 3.1 is done by induction on k, the number of cliques, and successive applications of the formula (1) and the well-known formula

$$\int_{\mathbb{R}^n} e^{-\pi(x_1^2 + \cdots + x_n^2)} \, dx_1 \cdots dx_n = 1. \tag{13}$$

3.2 We can derive local functional equations from Theorem 3.1 just as in §1.2. The result is as follows:

$$\begin{aligned} &\int_{\mathrm{Sym}_G^{*+}} H_G(\alpha, \beta; Y) \hat{f}(Y) \mu_G(dY) \\ &\qquad = \Gamma_G(\alpha, \beta) \sum_{p \in \mathrm{sgn}(G)} u_G^{(p)}(\alpha, \beta) \int_{\Omega_{G,p}} H_G(\alpha, \beta; \phi(X)) f(X) \, dX, \end{aligned}$$

where $\hat{f}$ is the Fourier transform of a rapidly decreasing function f on $\mathrm{Sym}_G(\mathbb{R})$ and

$$\begin{aligned} u_G^{(p)}(\alpha, \beta) \quad &= \quad (2\pi)^{\sum_C (p_C - q_C)\alpha_C - \sum_S (p_S - q_S)\nu(S)\beta_S} \\ &\qquad \times \exp\left(\left(\sum_C (p_C - q_C)\alpha_C - \sum_S (p_S - q_S)\nu(S)\beta_S \right) \frac{\pi\sqrt{-1}}{2} \right). \end{aligned}$$

3.3 Just as the local functional equation (3) can be generalized to (5), the local functional equation above can be generalized to functional equations of local zeta functions attached to $\Omega^*_{G,p}$ for arbitrary $p \in \mathrm{sgn}(G)$. For $p \in \mathrm{sgn}(G)$, we define the local zeta functions by

$$\Phi_{G,p}(\alpha,\beta;f) = \int_{\Omega_{G,p}} H(\alpha,\beta;\phi(X))f(X)\,dX, \quad (f \in \mathcal{S}(\mathrm{Sym}_G(\mathbb{R})),$$
$$\Phi^*_{G,p}(\alpha,\beta;f^*) = \int_{\Omega^*_{G,p}} H(\alpha,\beta;Y)f^*(Y)\,\mu_G(dY), \quad (f^* \in \mathcal{S}(\mathrm{Sym}^*_G(\mathbb{R})).$$

Then we can prove that the integral defining $\Phi^*_{G,p}(\alpha,\beta;f^*)$ is absolutely convergent under the conditions (1), (2) and (3) in Theorem 3.1. As is remarked in §3.1, $H(\alpha,\beta;\phi(X))$ can be expressed as

$$H(\alpha,\beta;\phi(X)) = |P_{G,0}(X)|^{\xi_0}\,|P_{G,1}(X)|^{\xi_1}\cdots|P_{G,k}(X)|^{\xi_k}$$

for some linear forms $\xi_0,\xi_1,\dots,\xi_k$ of $\alpha_1,\dots,\alpha_k,\beta_2$. Then the integral defining $\Phi_{G,p}(\alpha,\beta;f)$ is absolutely convergent for $\Re(\xi_0),\Re(\xi_1),\dots,\Re(\xi_k) > 0$. Moreover $\Phi_{G,p}(\alpha,\beta;x)$, $\Phi_{G,p}(\alpha,\beta;f^*)$ can be continued meromorphically to $\mathbb{C}^{k+1}$ as meromorphic functions of $\alpha_1,\dots,\alpha_k,\beta_2$.

Thoerem 3.2 *Assume that the condition* (1) *in Theorem 3.1 is satisfied and let* $\Gamma_G(\alpha,\beta)$ *be the same gamma factor as in Theorem* 3.1. *Then the following functional equations hold for any* $f \in \mathcal{S}(\mathrm{Sym}_G(\mathbb{R}))$*:*

$$\Phi^*_{G,p}(\alpha,\beta;\hat{f}) = \Gamma_G(\alpha,\beta)\sum_{q\in\mathrm{sgn}(G)} u_G^{(p.q)}(\alpha,\beta)\Phi_{G,q}(\alpha,\beta;f),$$

where $u_G^{(p.q)}(\alpha,\beta)$ $(p,q \in \mathrm{sgn}(G))$ *are meromorphic functions of* α,β *independent of* f.

The coefficients $u_G^{(p.q)}(\alpha,\beta)$ have elementary expressions in terms of Shintani's $u_{ij}^{(n)}$ (see (4)). However the explicit expression for general signature distributions p and q is too complicated to state here. The proof of the functional equation above is similar to the proof of Theorem 3.1 due to Letac and Massam. The necessary modification is to replace the formulas (1) and (13) in their argument by Shintani's functional equation (5) and the formula for the Fourier transforms of quadratic characters ([We])

$$\int_{\mathbb{R}^m} \exp(2\pi i\,{}^t xTx)\exp(2\pi i(x,y))\,dx$$
$$= 2^{-m/2}e^{i\pi(p-q)/4}|\det T|^{-1/2}\exp(-\frac{\pi i}{2}{}^t yT^{-1}y),$$

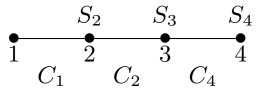
where (p,q) is the signature of a real non-degenerate symmetric matrix T of size m. Moreover, in the final stage of the proof, we have to appeal to the 2-variable version of Shintani's functional equation, namely, the local functional equations for the prehomogeneous vector space $(P_{r,n-r},\rho,\mathrm{Sym}(n))$, where

$$P_{r,n-r} = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \;\middle|\; A \in GL(r),\ B \in M(n-r,r),\ C \in GL(n-r) \right\},$$
$$\rho(g)X = gX\,{}^t g \ (g \in P_{r,n-r},\ X \in \mathrm{Sym}(n)).$$

Details of the proof will appear elsewhere.

3.4 As an example let us consider the non-homogeneous graph G of type A_4:

A_4 :



$$G = (V, E), \quad V = \{1, 2, 3, 4\},$$
$$E = \{\{1,1\}, \{2,2\}, \{3,3\}, \{4,4\}, \{1,2\}, \{2,3\}, \{3,4\}\}.$$

Then, the cliques and the separators in perfect order are given, for example, by

$$C_1 = \{1,2\}, \quad C_2 = \{2,3\}, \quad C_3 = \{3,4\}, \quad S_2 = \{2\}, \quad S_3 = \{3\}.$$

Then the polynomials we are interested in are the following:

$$Q_{C_1}(Y) = \det\begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}, \quad Q_{C_2}(Y) = \det\begin{pmatrix} y_3 & y_4 \\ y_4 & y_5 \end{pmatrix}, \quad Q_{C_3}(Y) = \det\begin{pmatrix} y_5 & y_6 \\ y_6 & y_7 \end{pmatrix},$$
$$Q_{S_2}(Y) = y_3, \quad Q_{S_3}(Y) = y_5,$$
$$P_{C_1}(X) = \frac{\det\begin{pmatrix} x_5 & x_6 \\ x_6 & x_7 \end{pmatrix}}{\det X}, \quad P_{C_2}(X) = \frac{x_1 x_7}{\det X}, \quad P_{C_3}(X) = \frac{\det\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}}{\det X},$$
$$P_{S_2}(X) = \frac{x_1 \det\begin{pmatrix} x_5 & x_6 \\ x_6 & x_7 \end{pmatrix}}{\det X}, \quad P_{S_3}(X) = \frac{x_7 \det\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}}{\det X}$$

for $Y = \begin{pmatrix} y_1 & y_2 & * & * \\ y_2 & y_3 & y_4 & * \\ * & y_4 & y_5 & y_6 \\ * & * & y_6 & y_7 \end{pmatrix} \in \mathrm{Sym}_G^*(\mathbb{R})$ and $X = \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ x_2 & x_3 & x_4 & 0 \\ 0 & x_4 & x_5 & x_6 \\ 0 & 0 & x_6 & x_7 \end{pmatrix} \in \mathrm{Sym}_G(\mathbb{R})$. The local functional equation given by Theorem 3.2 relates the Fourier transform of

$$\begin{aligned} &H_G(\alpha, \beta; \phi(X)) \\ &= \left(\frac{\left|\det\begin{pmatrix} x_5 & x_6 \\ x_6 & x_7 \end{pmatrix}\right|}{|\det X|}\right)^{\alpha_1} \left(\frac{|x_1|\,|x_7|}{|\det X|}\right)^{\alpha_2} \\ &\quad \times \left(\frac{\left|\det\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}\right|}{|\det X|} \Big/ \frac{|x_7|\left|\det\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}\right|}{|\det X|}\right)^{\alpha_3} \left(\frac{|x_1|\left|\det\begin{pmatrix} x_5 & x_6 \\ x_6 & x_7 \end{pmatrix}\right|}{|\det X|}\right)^{-\beta} \\ &= |x_1|^{\alpha_2 - \beta}\, |x_7|^{\alpha_2 - \alpha_3} \left|\det\begin{pmatrix} x_5 & x_6 \\ x_6 & x_7 \end{pmatrix}\right|^{\alpha_1 - \beta} |\det X|^{-(\alpha_1 + \alpha_2 - \beta)} \end{aligned} \tag{14}$$

to

$$\begin{aligned} &H_G\left(\alpha - \frac{c+1}{2}, \beta - \frac{s+1}{2}; Y\right) \\ &= \left|\det\begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}\right|^{\alpha_1 - \frac{3}{2}} \left|\det\begin{pmatrix} y_3 & y_4 \\ y_4 & y_5 \end{pmatrix}\right|^{\alpha_2 - \frac{3}{2}} \frac{\left|\det\begin{pmatrix} y_5 & y_6 \\ y_6 & y_7 \end{pmatrix}\right|^{\alpha_3 - \frac{3}{2}}}{|y_3|^{\alpha_3 - \frac{1}{2}}} |y_5|^{-\beta + \frac{1}{2}}. \end{aligned}$$

Let us consider the subgroup $\mathcal{G}$ of linear transformations on $\mathrm{Sym}_G(\mathbb{R})$ leaving the polynomials x_1, x_7, $\det\begin{pmatrix} x_5 & x_6 \\ x_6 & x_7 \end{pmatrix}$, $\det X$ relatively invariant. Then the group $\mathcal{G}$ is given by

$$\mathcal{G} = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v & t_2 & 0 & 0 & 0 & 0 & 0 \\ \frac{v^2}{t_1} & \frac{2t_2v}{t_1} & t_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_5 & \frac{2t_6u}{t_7} & \frac{u^2}{t_7} \\ 0 & 0 & 0 & 0 & 0 & t_6 & u \\ 0 & 0 & 0 & 0 & 0 & 0 & t_7 \end{pmatrix} \;\middle|\; t_1t_3 = t_2^2,\ t_5t_7 = t_6^2,\ t_2^2t_6^2 = t_1t_4^2t_7 \right\}.$$

Since $\mathcal{G}$ is a group of dimension 6 acting on the 7-dimensional vector space Sym_G, this can not be a prehomogeneous vector space. Thus, in this case, Theorem 3.2 gives a local functional equation which is not obtained from the theory of prehomogeneous vector spaces.

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